Name________________________ Teacher________________________

# GOSFORD HIGH SCHOOL

# MATHEMATICS

# EXTENSION 2

# HSC

# 2016

# ASSESSMENT TASK 2

**General Instructions**

- Reading Time – 5 minutes
- Working Time – 90 minutes
- Write on one side of your paper only.
- Start a new page for each question.
- Write your name on each page you submit.
- Correct setting out ***must*** be shown or full marks may not be awarded.
- Board approved calculators may be used.
- A Reference sheet is provided.

**Total Marks – 51**

**Section 1 (6 marks)**
Questions 1- 6

**Section 2 (45 marks)**
Questions 7 - 9

**SECTION 1** - **(6 marks)** Answer these questions on the multiple choice answer sheet.

1. If ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^7$ is equal to:

A. $128\omega$ B. $-128\omega$

C. $128\omega^2$ D. $-128\omega^2$

2. If $\text{Re}\left(\dfrac{z-8i}{z+6}\right)=0$, then $z$ lies on the curve:

A. $x^2+y^2+6x-8y=0$ B. $4x-3y+24=0$

C. $x^2+y^2-8=0$ D. none of these

3. A polynomial P(x) of fourth degree with real coefficients has the following properties:

$P(1)=0, P'(1)\neq 0$
$P(2)\neq 0, P'(2)=P''(2)=0$

What is the greatest number of complex non-real roots the polynomial could have?

A. 0 B. 1

C. 2 D. 3

4. The cubic equation $2y^3-9y^2+12y+k=0$ has two equal roots.

What are the possible values for $k$?

A. −4 and −5 B. −4 and 5

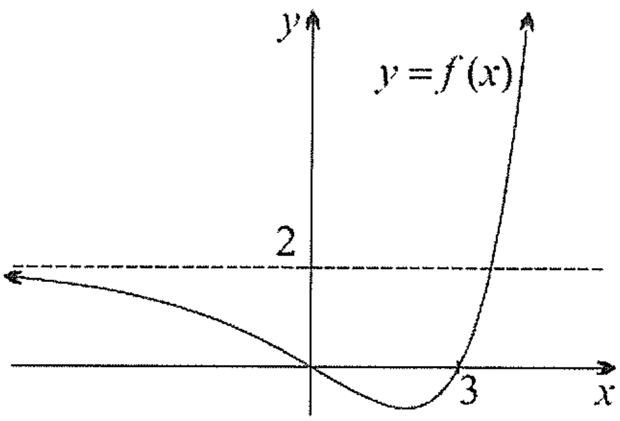
C. 4 and −5 D. 4 and 5

5. By de Moivre's theorem, the value of $(1+i)^{10}$ is:

A. purely real B. purely imaginary

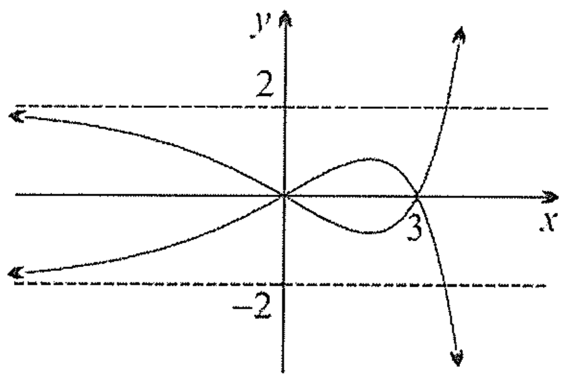
C. a real multiple of $(1+i)$ D. an imaginary multiple of $(1+i)$

6. The graph of $y=f(x)$ is shown below.

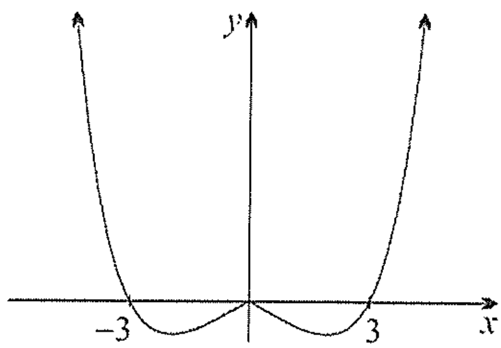



Which is the correct graph of $|y|=f(x)$ ?

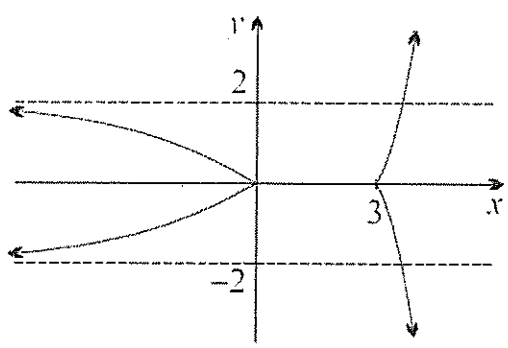
A.



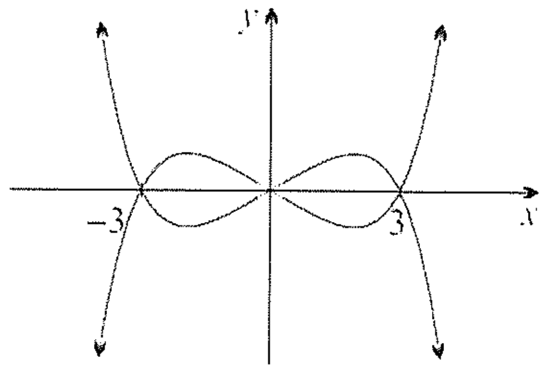

B.




C.




D.

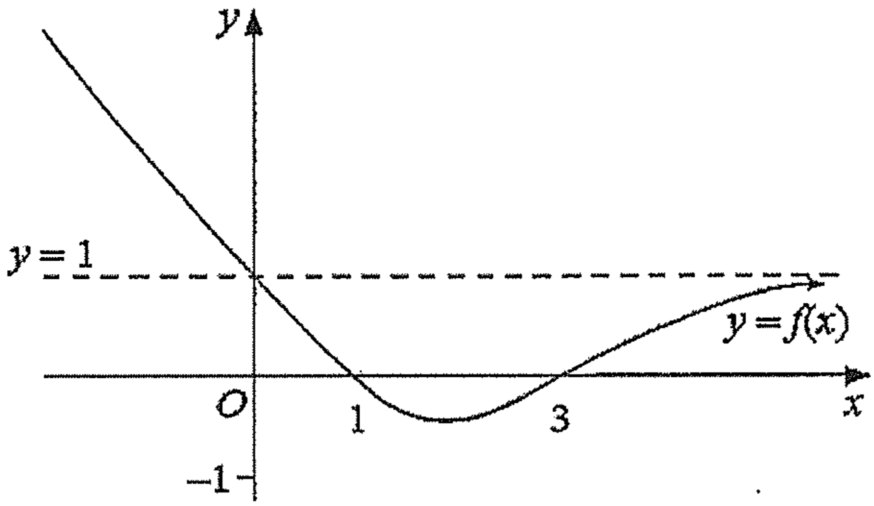


**SECTION 2** - **(45 marks)** Answer each question on a separate page

**Question 7 (15 marks) Start a new page** **Marks**

a) Give the equation(s) of all asymptotes on the graph of $f(x)=\dfrac{x^2-x-6}{x-1}$. **1**

b)




The diagram shows the graph of the function $y=f(x)$.
The function has a horizontal asymptote at y=1.
Draw separate half page sketches of the graphs of the following:

i) $y^2=f(x)$ **2**

ii) $y=f(|x|)$ **2**

iii) $y=\dfrac{1}{f(x)}$ **2**

c) Find the point of intersection of the curves $x^2-y^2=c^2$ and $xy=c^2$ and prove that the curves meet at right angles. **3**

d) i) Show whether the function $f(x)=2|x-1|-|x|+2|x+1|$ is even, odd or neither, giving reasons. **2**

ii) Sketch the graph of the function $f(x)=2|x-1|-|x|+2|x+1|$, clearly showing all intercepts with the coordinate axes and critical points. Label all branches with the relevant equations. **3**

**Question 8 (15 marks) Start a new page** **Marks**

a) Find real numbers $a$ and $b$ such that $\dfrac{1}{x(\pi-2x)}=\dfrac{a}{x}+\dfrac{b}{\pi-2x}$. **1**

b) The polynomial $P(x)$ leaves a remainder of 9 when divided by $(x-2)$ and a remainder of 4 when divided by $(x-3)$. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$. **2**

c) Given that $x+i$ is a factor of $P(x)=x^4+3x^3+6x^2+3x+5$, factorise $P(x)$ over the complex plane. **3**

d) Determine all the roots of $8x^4-25x^3+27x^2-11x+1=0$ given that it has a root of multiplicity 3. **3**

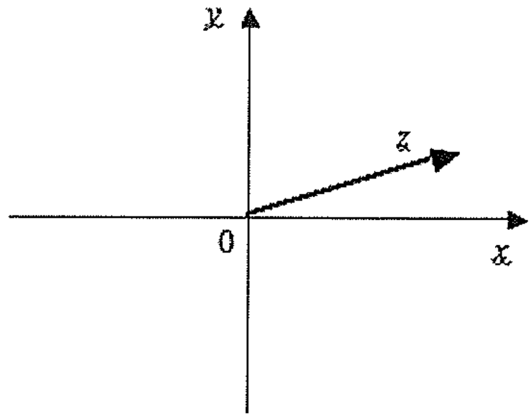
e) The equation $x^3+px+1=0$ has three real non-zero roots $\alpha$, $\beta$ and $\gamma$.

i) Find the values of $\alpha^2+\beta^2+\gamma^2$ and $\alpha^4+\beta^4+\gamma^4$ in terms of $p$. **3**

ii) Find the monic equation, with co-efficients in terms of p, whose roots are $\dfrac{\alpha}{\beta\gamma},\dfrac{\beta}{\alpha\gamma},\dfrac{\gamma}{\alpha\beta}$. **3**

**Question 9 (15 marks) Start a new page**

a)



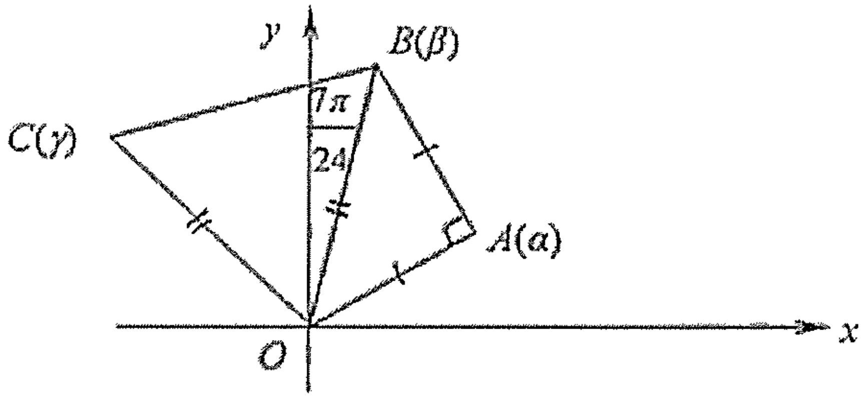

i) Copy this Argand diagram and draw vectors for $\bar{z}$, $iz$ and $\bar{z}-iz$ on the same diagram. **2**
Show carefully any details about relative lengths and directions.

ii) Show that if $z$ has argument $\theta$, $0<\theta<\dfrac{\pi}{2}$ and modulus $r$, then **2**

$$\left|\bar{z}-iz\right|=2r\left|\sin\left(\theta+\frac{\pi}{4}\right)\right|$$

iii) Find the value of $\theta$ for which $\left|\bar{z}-iz\right|$ takes its maximum value. **1**

b) Points $A$, $B$ and $C$ represent the complex numbers $\alpha$, $\beta$ and $\gamma$ in the Argand Diagram respectively.




*Not to scale*

$\Delta OAB$ is right isosceles at $A$, $\Delta COB$ is isosceles with $OB=OC$ and $\angle OBC=\dfrac{7\pi}{24}$.

i) Copy or trace the diagram onto your writing booklet and find the value of $\angle AOC$. **1**

ii) Explain why $\gamma=\sqrt{2}\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)\alpha$ **2**

iii) Hence find the value of $2\alpha^2+\gamma^2+\alpha\gamma\sqrt{2}$. **2**

c) i) Use De Moivre's theorem to show that **2**

$\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$

and $\sin 5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$

Hence express $\tan 5\theta$ in terms of $\tan\theta$.

ii) Hence show that $\tan\dfrac{\pi}{5}\tan\dfrac{2\pi}{5}\tan\dfrac{3\pi}{5}\tan\dfrac{4\pi}{5}=5$. **3**

## <Course Name>

## Student Name/Number:________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

| Sample: | 2 + 4 = | (A) 2 | (B) 6 | (C) 8 | (D) 9 |
|---|---|---|---|---|---|
| | | A ◯ | B ⬤ | C ◯ | D ◯ |

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ⬤ B ⬤ C ◯ D ◯

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ⬤ B ⬤ C ◯ D ◯

1. A ◯ B ◯ C ◯ D ◯
2. A ◯ B ◯ C ◯ D ◯
3. A ◯ B ◯ C ◯ D ◯
4. A ◯ B ◯ C ◯ D ◯
5. A ◯ B ◯ C ◯ D ◯
6. A ◯ B ◯ C ◯ D ◯

$\omega^3 = 1 \Rightarrow \omega^3 - 1 = 0$

$\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1)$

$\therefore \omega^2 + \omega + 1 = 0$

$\omega + 1 = -\omega^2$

$\therefore (1 + \omega - \omega^2)^7$

$= (-\omega^2 - \omega^2)^7$

$= (-2\omega^2)^7$

$= -128\omega^{14}$

$= -128(\omega^3)^4 \omega^2$

$= -128\omega^2$ (D)

---

$Re\left(\frac{z-8i}{z+6}\right) = 0$

$\frac{z-8i}{z+6} = \frac{x+i(y-8)}{(x+6)+iy} \times \frac{(x+6)-iy}{(x+6)-iy}$

$Re\left(\frac{z-8i}{z+6}\right) = \frac{x(x+6) - iy(i(y-8))}{(x+6)^2 + y^2}$

$\therefore x^2 + 6x + y^2 - 8y = 0$ (A)

---

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$P(1) = 0$, $P'(1) \neq 0$ } single root → not complex

$P(2) \neq 0$ – not root

$P'(2) = P''(2) = 0$ ← poss. horiz pt inflex

+ complex roots in conjugate pairs $\therefore 2$ (C)

---

$2y^3 - 9y^2 + 12y + k = 0$

$P'(y) = 6y^2 - 18y + 12$

roots are $\alpha, \alpha, \beta$

$\therefore 2\alpha + \beta = \frac{9}{2} \Rightarrow \beta = \frac{9}{2} - 2\alpha$

$\alpha^2 + 2\alpha\beta = 6$

$\alpha^2 + 2\alpha\left(\frac{9}{2} - 2\alpha\right) = 6$

$\alpha^2 + 9\alpha - 4\alpha^2 = 6$

$\alpha^2 - 3\alpha + 2 = 0$

$(\alpha - 2)(\alpha - 1) = 0$

$\alpha = 2, 1$

$\alpha = 2$, $\beta = \frac{1}{2}$, $\alpha\beta\gamma = 2$

$\alpha = 1$, $\beta = 2\frac{1}{2}$, $\alpha\beta\gamma = 5/2$

$\therefore \alpha\beta\gamma = \frac{-k}{2}$

$\therefore -k = 4, 5$

$k = -4, -5$ (A)

---

5) $(1+i)^{10} = \left(\sqrt{2} \text{ cis } \frac{\pi}{4}\right)^{10}$

$= 2^5 \text{ cis } \frac{5\pi}{2}$

$= 2^5 \text{ cis } \frac{\pi}{2}$

$= 2^5 (0 + i)$ (B)

---

6) C

---

## Question 7

a) $f(x) = \frac{x^2 - x - 6}{x - 1}$

$= \frac{x(x-1) - 6}{x - 1}$

$= x - \frac{6}{x-1}$

$\therefore$ asymptotes at: $x = 1$, $y = x$ [1]

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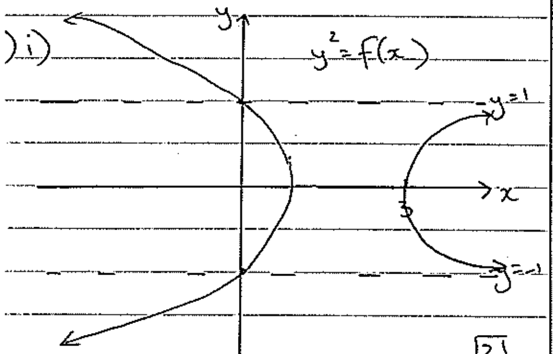
b) i)




① reflection, ① shape [2]

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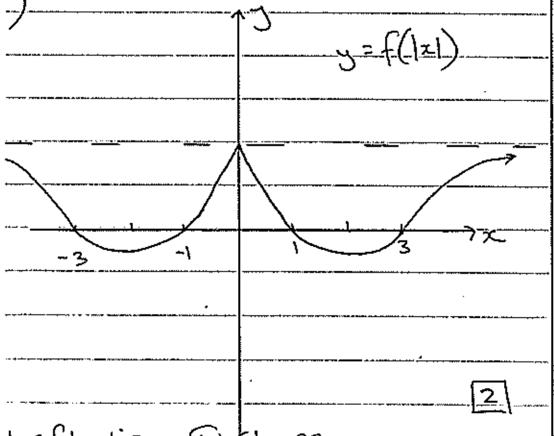
ii)




① reflection, ① shape [2]

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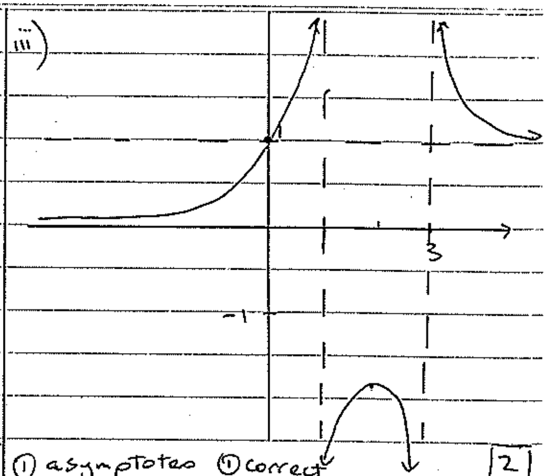
iii)




① asymptotes ① correct [2]

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c) $x^2 - y^2 = c^2$ ①

$xy = c^2$ ②

m of tangent on $x^2 - y^2 = c^2$

$2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$

Must not use general point.

m of tangent on $xy = c^2$

$y + x\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-y}{x}$ ① $m_1, m_2$

for pt int. $\frac{x_1}{y_1} \times \frac{-y_1}{x_1} = -1$

② $y = \frac{c^2}{x}$

① $x^2 - \frac{c^4}{x^2} = c^2$

$x^4 - c^2x^2 - c^4 = 0$.

$x^2 = c^2 \pm \sqrt{c^4 + 4c^4}$

$= \frac{c^2 \pm c^2\sqrt{5}}{2}$ ①

$x = c\sqrt{\frac{1+\sqrt{5}}{2}}$ | $x = c\sqrt{\frac{1-\sqrt{5}}{2}}$

$y = \frac{c\sqrt{2}}{\sqrt{1+\sqrt{5}}}$ | $y = \frac{c\sqrt{2}}{\sqrt{1-\sqrt{5}}}$

& $m_1 m_2 = \frac{x}{y} \cdot \frac{-y}{x}$

$\therefore = -1$ (1)

$\therefore$ curves meet at rt $\angle$'s [3]

) $f(x) = 2|x-1| - |x| + 2|x+1|$

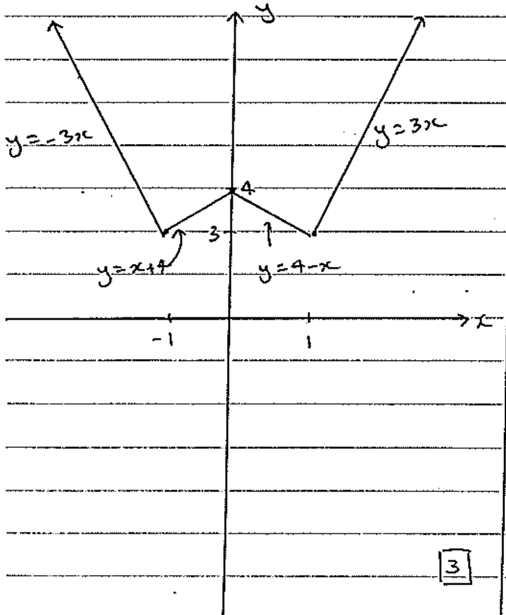
$f(-x) = 2|(-x-1)| - |-x| + 2|-x+1|$

$= 2|-(x+1)| - x + 2|-(x-1)|$ (1)

$= 2|x+1| - x + 2|x-1|$

$= f(x)$

$\therefore$ even (1) [2]




[3]

) shape + points

) labels

Question 8

) $a(\pi - 2x) + bx \equiv 1$

$-2a + b = 0$

$a\pi = 1$

$a = \frac{1}{\pi}$

$-\frac{2}{\pi} + b = 0$ correct

$b = \frac{2}{\pi}$ [1]

) $P(2) = 9$

$P(3) = 4$

$P(x) = (x-2)(x-3) + ax + b$ (1)

$P(2) = 2a + b = 9$

$P(3) = 3a + b = 4$

$-a = 5$

$a = -5$

$b = +19$

$\therefore$ remainder $= -5x + 19$ (1) [2]

) $(x+i)$ & $(x-i)$ factors

real coeff $\rightarrow$ conjugate pairs

$\therefore x^2+1$ is factor (1)

$$\begin{array}{r} x^2 + 3x + 5 \\ x^2+1 \;\big)\; x^4 + 3x^3 + 6x^2 + 3x + 5 \\ x^4 + x^2 \\ \hline 3x^3 + 5x^2 \\ 3x^3 + 3x \\ \hline 5x^2 - 3x \\ 5x^2 + 5 \\ \hline 0 \end{array}$$

$P(x) = (x+i)(x-i)(x^2+3x+5)$ (1)

roots of $x^2+3x+5$

$x = \frac{-3 \pm \sqrt{9-20}}{2}$

$= \frac{-3 \pm i\sqrt{11}}{2}$

$\therefore P(x) = (x-i)(x+i)\left(x - \left(\frac{-3+i\sqrt{11}}{2}\right)\right)\left(x - \left(\frac{-3-i\sqrt{11}}{2}\right)\right)$

$= \frac{1}{4}(x-i)(x+i)(2x+3-i\sqrt{11})$ (1) $(2x+3+i\sqrt{11})$ [3]

d) $P(x) = 8x^4 - 25x^3 + 27x^3 - 11x + 1$

$P'(x) = 32x^3 - 75x^2 + 54x - 11$

$P''(x) = 96x^2 - 150x + 54$

mult 3 $\rightarrow$ $P(x) = P'(x) = P''(x) = 0$

solve

$96x^2 - 150x + 54 = 0$ (1)

$16x^2 - 25x + 9 = 0$

$x = \frac{25 \pm \sqrt{49}}{32}$

$= 1, \frac{9}{16}$

check $P(1) = 8 - 25 + 27 - 11 = 0$

$P'(1) = 32 - 75 + 54 - 11 = 0$

$\therefore (x-1)^3$ is factor

$\therefore P(x) = (x-1)^3(ax+b)$

$P(0) = (-1)^3 b$ $P(0) = 0-0+0-0+1$

& $\therefore -b = 1$

$\therefore b = -1$ (1)

check coeff $x^4 \rightarrow 8$

$\therefore ax^4 = 8x^4$

$a = 8$

$\therefore$ Roots are $1, 1, 1, \frac{1}{8}$ (1) answer

as $P(x) = (x-1)^3(8x-1)$ [3]

---

) $x^3 + px + 1 = 0$

) poly with roots $\alpha^2, \beta^2, \gamma^2$

$x^{3/2} + px^{1/2} + 1 = 0$.

$(x^{3/2} + px^{1/2})^2 = (-1)^2$

$x^3 + 2px^2 + p^2x = 1$

$x^3 + 2px^2 + p^2x - 1 = 0$.

$\therefore \alpha^2 + \beta^2 + \gamma^2 = -2p$ (1)

ow new poly roots $\alpha^4, \beta^4, \gamma^4$

$x^{3/2} + 2px + p^2x^{1/2} - 1 = 0$

$(x^{3/2} + p^2x^{1/2})^2 = (1 - 2px)^2$

$x^3 + 2p^2x^2 + p^4x = 1 - 4px + 4p^2x^2$

$x^3 - 2p^2x^2 + x(p^4 + 4p) - 1 = 0$

$\therefore \alpha^4 + \beta^4 + \gamma^4 = 2p^2$ (2)

[3]

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Roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma}, \frac{\gamma}{\alpha\beta}$

$\therefore \frac{\alpha^2}{\alpha\beta\gamma}, \frac{\beta^2}{\alpha\beta\gamma}, \frac{\gamma^2}{\alpha\beta\gamma}$ (1)

$\alpha\beta\gamma = -1$

$\therefore$ roots $-\alpha^2, -\beta^2, -\gamma^2$

$\therefore \Sigma\alpha_1 = -(\Sigma\alpha^2)$

$= 2p$

$\Sigma\alpha_1\beta_1 = \Sigma\alpha^2\beta^2$

$= p^2$ (1)

$\alpha_1\beta_1\gamma_1 = -\alpha^2\beta^2\gamma^2$

$= -1$

$\therefore$ eqn is

$x^3 - 2px + p^2x + 1 = 0$ (1) [3]

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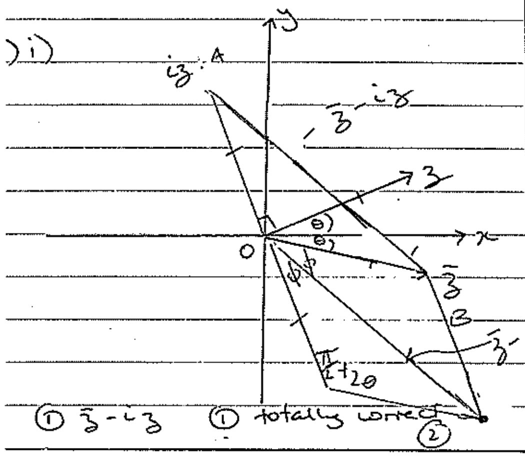
Question 9

a) i)




(1) $z - iz$ (1) totally correct (2)

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) $\angle AOB = \frac{\pi}{2} + 2\theta$ $\quad z = r\,\text{cis}\,\theta$

$|z - iz|^2 = r^2 + r^2 - 2r^2\cos(\frac{\pi}{2} + 2\theta)$

(cosine rule)

$RHS = 2r^2(1 - \cos(\frac{\pi}{2} + 2\theta))$ (1)

$\cos 2\theta = 1 - 2\sin^2\theta$

$\therefore \cos(\frac{\pi}{2} + 2\theta) = 1 - 2\sin^2(\frac{\pi}{4} + \theta)$

$RHS = 2r^2(1 - (1 - 2\sin^2(\frac{\pi}{4} + \theta)))$

$= 2r^2(2\sin^2(\frac{\pi}{4} + \theta))$

$= 4r^2\sin^2(\frac{\pi}{4} + \theta)$

$\therefore |z - iz| = \sqrt{4r^2\sin^2(\frac{\pi}{4} + \theta)}$

$= 2r|\sin(\frac{\pi}{4} + \theta)|$ (1)

[2]

) max value at $(\frac{\pi}{4} + \theta) = \frac{\pi}{2}$

$\therefore \theta = \frac{\pi}{4}$

[1]

---

b) i) $\angle AOC = (\pi - \frac{14\pi}{24} + \frac{\pi}{4})$

$= \frac{2\pi}{3}$ [1]

ii) let $\alpha = r\,\text{cis}\,\theta$

$|\beta| = \sqrt{r^2 + r^2}$

$= r\sqrt{2}$

$\arg(\beta) = \theta + \frac{\pi}{4}$

$\gamma = \beta(\text{cis}(\pi - \frac{14\pi}{24}))$ (1)

$= \beta\,\text{cis}\,\frac{5\pi}{12}$

$= r\sqrt{2}\,\text{cis}(\theta + \frac{\pi}{4})\,\text{cis}(\frac{5\pi}{12})$

$= r\sqrt{2}\,\text{cis}(\theta + \frac{\pi}{4} + \frac{5\pi}{12})$

$= r\sqrt{2}\,\text{cis}(\theta + \frac{2\pi}{3})$

$= \sqrt{2}(r\,\text{cis}\,\theta\,\text{cis}\,\frac{2\pi}{3})$

$= \sqrt{2}(\text{cis}\,\frac{2\pi}{3})\alpha$ (1)

[2]

iii) $2\alpha^2 + \gamma^2 + \alpha\gamma\sqrt{2}$

$= 2\alpha^2 + (\sqrt{2}\alpha\,\text{cis}\,\frac{2\pi}{3})^2$

$+ \alpha\sqrt{2}(\sqrt{2}\alpha\,\text{cis}\,\frac{2\pi}{3})$

$= 2\alpha^2 + 2\alpha^2\text{cis}\,\frac{4\pi}{3} + 2\alpha^2\text{cis}\,\frac{2\pi}{3}$

$= 2\alpha^2 + 2\alpha^2\text{cis}(\frac{-2\pi}{3}) + 2\alpha^2\text{cis}\,\frac{2\pi}{3}$

$= 2\alpha^2 + 2\alpha^2(\cos(\frac{-2\pi}{3}) + i\sin(\frac{-2\pi}{3})$

$+ \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}))$ (1)

$= 2\alpha^2 + 2\alpha^2(2\cos\frac{2\pi}{3})$

$= 2\alpha^2(1 + 2(-\frac{1}{2}))$

$= 2\alpha^2(1 - 1)$

$= 0$ (1)

[2]

i) $(\text{cis}\,\theta)^5 = \cos 5\theta + i\sin 5\theta$

$= c^5 + i5c^4s - 10c^3s^2 - 10ic^2s^3$

$+ 5cs^4 + is^5$

$\therefore \cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$ (1)

(equate real parts)

$\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$

$\therefore \tan 5\theta = \dfrac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$

$= \dfrac{\frac{5c^4s}{c^5} - \frac{10c^2s^3}{c^5} + \frac{s^5}{c^5}}{\frac{c^5}{c^5} - \frac{10c^3s^2}{c^5} + \frac{5cs^4}{c^5}}$

(1)

$= \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

[2]

) 5 roots of $t^5 - 10t^3 + 5t = 0$

ie $\tan 5\theta = 0$

but only 4 roots

$\therefore t(t^4 - 10t^2 + 5) = 0$ (1)

$\therefore$ solve $t^4 - 10t^2 + 5 = 0$

$\alpha\beta\gamma\delta = 5$ (1)

if $\tan 5\theta = 0$.

$\therefore 5\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

$\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$ (1)

but only require roots

$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$

[3]